

# Digital Image Processing and Pattern Recognition

E1528

Fall 2022-2023

Lecture 7



## Filtering in the Frequency Domain

**INSTRUCTOR**

DR / AYMAN SOLIMAN

## ➤ Contents

- A Brief History of the Fourier Series and Transform
- Fourier Series
- Sampling and the Fourier Transform of Sampled Functions
- Function Reconstruction (Recovery) from Sampled Data
- Image Smoothing Using Lowpass Frequency Domain Filter



## ➤ Objectives of Filtering in the Frequency Domain

- Understand the **meaning** of **frequency** domain filtering, and how it **differs** from filtering in the **spatial** domain.
- Be familiar with the concepts of **sampling**, **function reconstruction**, and **aliasing**.
- Understand **convolution** in the **frequency** domain, and how it is related to filtering.
- Know how to **obtain frequency domain filter functions** from **spatial kernels**, and **vice versa**.
- Be able to construct **filter transfer functions** directly in the frequency domain.

## ➤ Objectives of Filtering in the Frequency Domain

- Understand why image **padding** is important.
- Know the **steps required to perform filtering** in the frequency domain.
- Understand when **frequency** domain filtering is **superior** to filtering in the **spatial** domain.
- Be familiar with **other filtering techniques** in the frequency domain, such as **unsharp masking** and **homomorphic** filtering.
- Understand the **origin** and **mechanics** of the **fast Fourier transform**, and how to use it effectively in image processing.

## ➤ Introduction

- **Filter**: A device or material for **suppressing** or **minimizing** waves or oscillations of **certain frequencies**.
- **Frequency**: The number of times that a periodic function **repeats** the same sequence of values during a unit variation of the independent variable.
- After a brief historical introduction to the **Fourier transform** and its **importance** in image processing, we start from basic principles of function **sampling**, and proceed step-by-step to derive the **one- and two-dimensional discrete Fourier transforms**.

## ➤ Introduction

- Together with **convolution**, the **Fourier transform** is a staple of frequency-domain processing.
- During this development, we also touch upon **several important aspects of sampling**, such as **aliasing**, whose treatment requires an understanding of the frequency domain and thus are best covered in this part.
- This material is followed by a formulation of filtering in the **frequency** domain, **paralleling** the **spatial** filtering techniques discussed previously.

## ➤ Introduction

- We conclude the derivation of the equations underlying the **fast Fourier transform (FFT)** and discuss its computational **advantages**.
- These advantages make **frequency-domain** filtering **practical** and, in many instances, **superior** to filtering in the **spatial** domain.

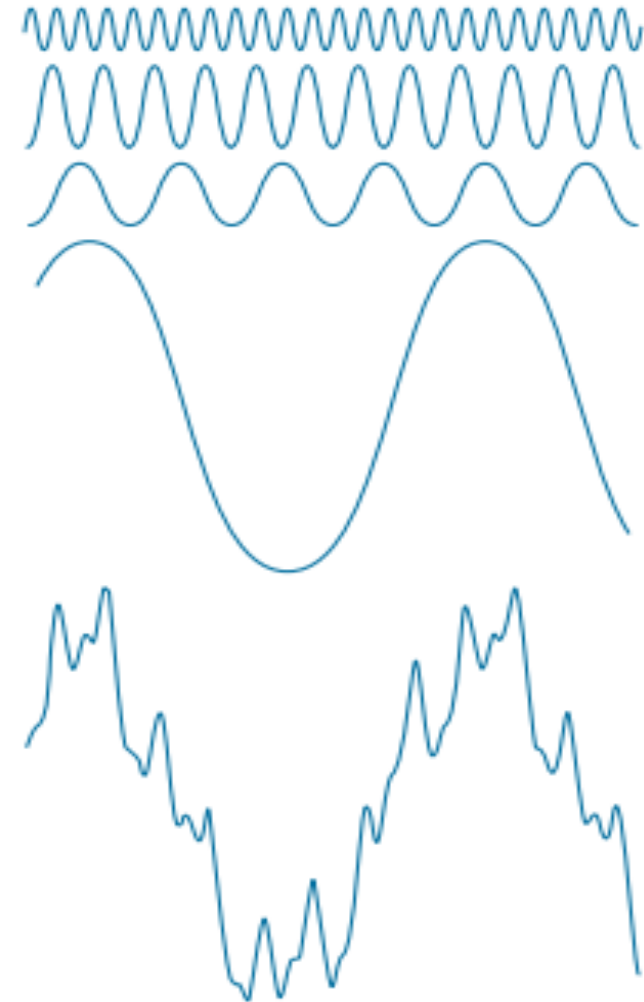
## ➤ A Brief History of the Fourier Series and Transform

- The French mathematician **Jean Baptiste Joseph Fourier** was born in 1768 in the town of Auxerre
- Basically, Fourier's contribution in this field states that any **periodic function** can be expressed as the **sum of sines and/or cosines of different frequencies**, each multiplied by a different **coefficient** (we now call this sum a Fourier series).
- It does not matter how **complicated** the function is; if it is periodic and satisfies some mild mathematical conditions, it can be represented by such a sum.



## ➤ A Brief History of the Fourier Series and Transform

- The function at the **bottom** is the **sum** of the **four** functions **above** it.
- Fourier's idea in 1807 that periodic functions could be represented as a weighted **sum of sines and cosines**.



## ➤ A Brief History of the Fourier Series and Transform

- The **initial** application of **Fourier's ideas** was in the field of **heat diffusion**, where they allowed formulation of **differential equations** representing heat flow in such a way that solutions could be obtained for the first time.
- During the **past century**, and especially in the **past 60 years**, entire industries and academic disciplines have succeeded as a result of Fourier's initial ideas.
- The advent of **digital computers** and the “discovery” of a **fast Fourier transform (FFT) algorithm** in the early 1960s revolutionized the field of signal processing.

## ➤ A Brief History of the Fourier Series and Transform

- As you learned in last lecture, it takes on the order of  $MNmn$  operations (multiplications and additions) to filter an  $M \times N$  image with a kernel of size  $m \times n$  elements.
- If kernel is **separable**, the number of operations is reduced to  $MN(m+n)$ .
- In next Section, you will learn that it takes on the order of  $2MN \log_2(MN)$  operations to perform the equivalent filtering process in the frequency domain, where the **2** in front arises from the fact that we have to compute a **forward** and an **inverse FFT**.

## ➤ Fourier Series

- a function  $f(t)$  of a continuous variable,  $t$ , that is periodic with a period,  $T$ , can be expressed as the **sum** of **sines** and **cosines** multiplied by appropriate coefficients. This sum, known as a Fourier series, has the form

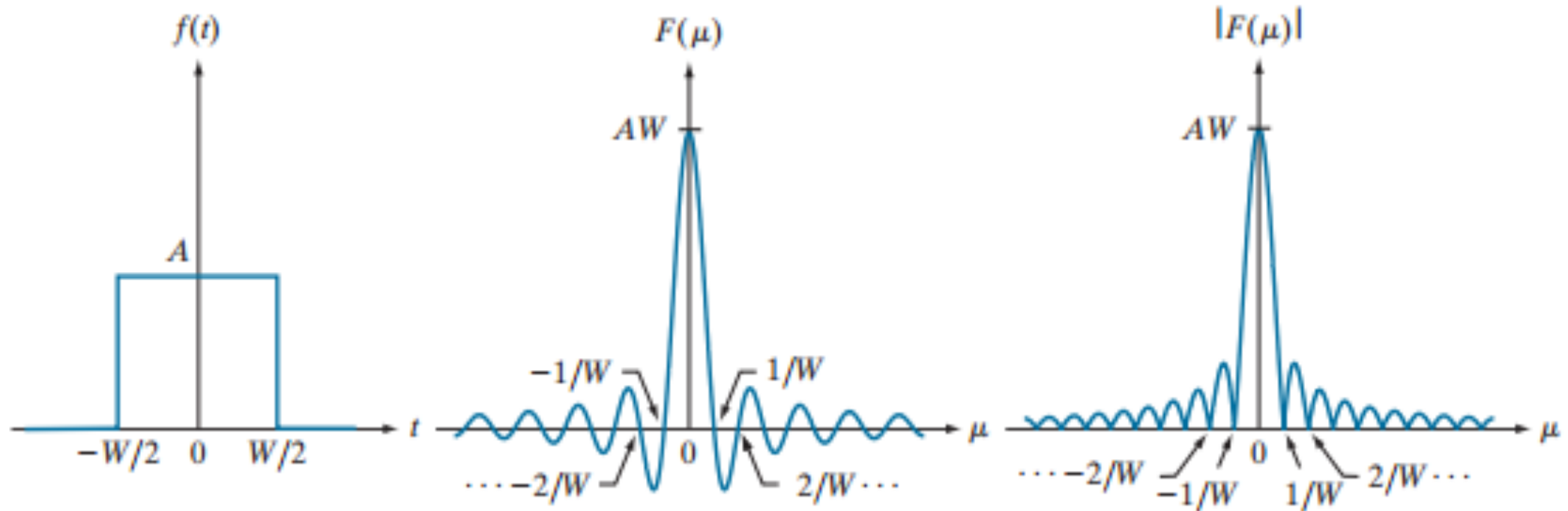
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

where  $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt$   $n = 0, \pm 1, \pm 2, \dots$

➤ **Obtaining the Fourier Transform of a Simple Continuous Function**

$$\begin{aligned} F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt \\ &= \frac{-A}{j2\pi\mu} \left[ e^{-j2\pi\mu t} \right]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu} \left[ e^{-j\pi\mu W} - e^{j\pi\mu W} \right] \\ &= \frac{A}{j2\pi\mu} \left[ e^{j\pi\mu W} - e^{-j\pi\mu W} \right] \\ &= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} \end{aligned}$$

## ➤ Obtaining the Fourier Transform of a Simple Continuous Function



a b c

(a) A box function, (b) its Fourier transform, and (c) its spectrum.

All functions extend to infinity in both directions.

## ➤ **Sampling and the Fourier Transform of Sampled Functions**

- In this section, we use the concepts from previous Section to formulate a **basis** for expressing sampling mathematically.
- Starting from basic principles, this will lead us to the Fourier transform of **sampled** functions. That is, the **discrete Fourier transform**.

## ➤ Sampling

- Continuous functions must be converted into a sequence of **discrete** values **before** they can be **processed** in a computer.
- This requires **sampling** and **quantization**.
- Consider a continuous function, **f(t)**, that we wish to sample at uniform intervals, **ΔT**, of the independent variable **t**.
- We assume initially that the function extends from **-∞ to ∞** with respect to **t**.  
One way to model sampling is to multiply **f(t)** by a sampling function **equal** to a train of impulses **ΔT** units apart.



## ➤ Sampling

- That is,

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

- where  $\tilde{f}(t)$  denotes the **sampled** function.
- Each component of this summation is an impulse **weighted** by the value of  $f(t)$  at the location of the impulse.
- The value of each sample is given by the “**strength**” of the weighted impulse, which we obtain by integration.

## ➤ Sampling

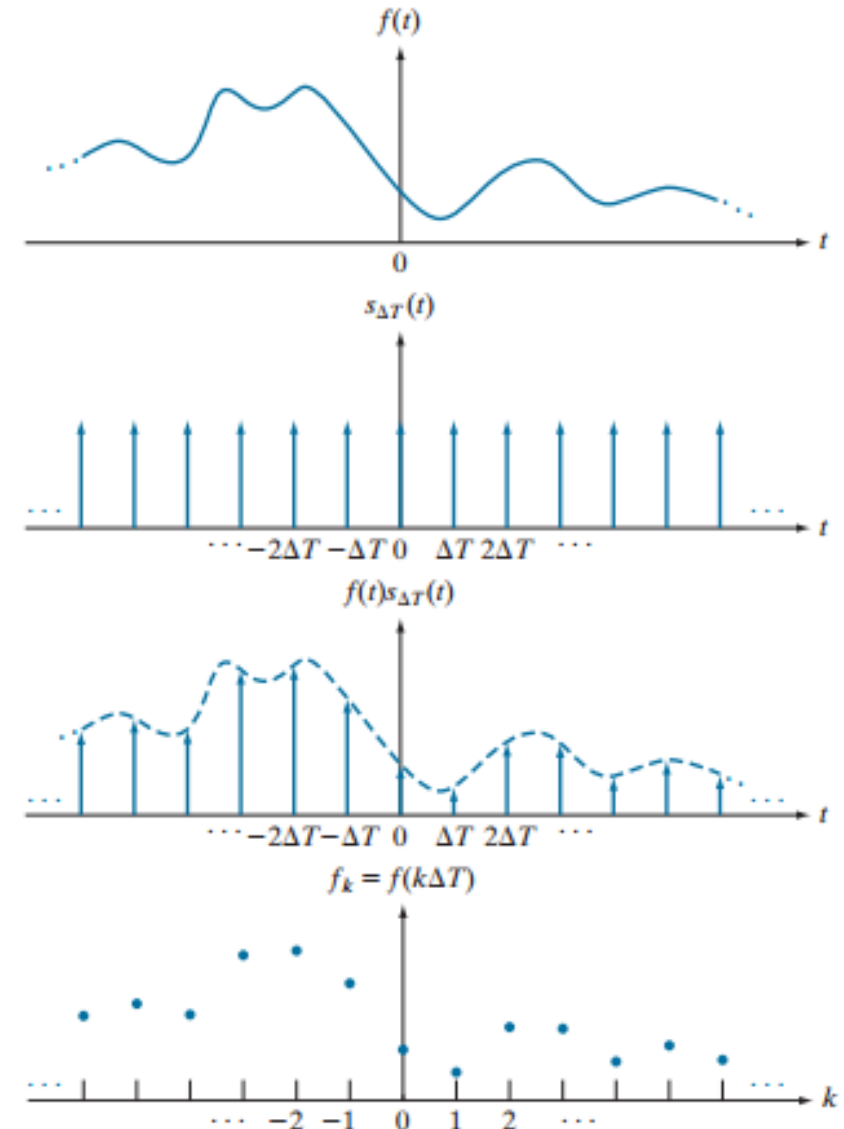
- That is, the value,  $f_k$ , of an arbitrary sample in the sampled sequence is given by

$$\begin{aligned} f_k &= \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt \\ &= f(k\Delta T) \end{aligned}$$

- where we used the **sifting property of  $\delta$** . This Equation holds for any integer value  $k = \dots, -2, -1, 0, 1, 2, \dots$  next Figure(d) shows the result, which consists of equally spaced samples of the original function.

## ➤ Sampling

- (a) A continuous function.
- (b) Train of impulses used to model sampling.
- (c) Sampled function formed as the product of (a) and (b).
- (d) Sample values obtained by integration and using the sifting property of impulses. (The dashed line in (c) is shown for reference. It is not part of the data.)

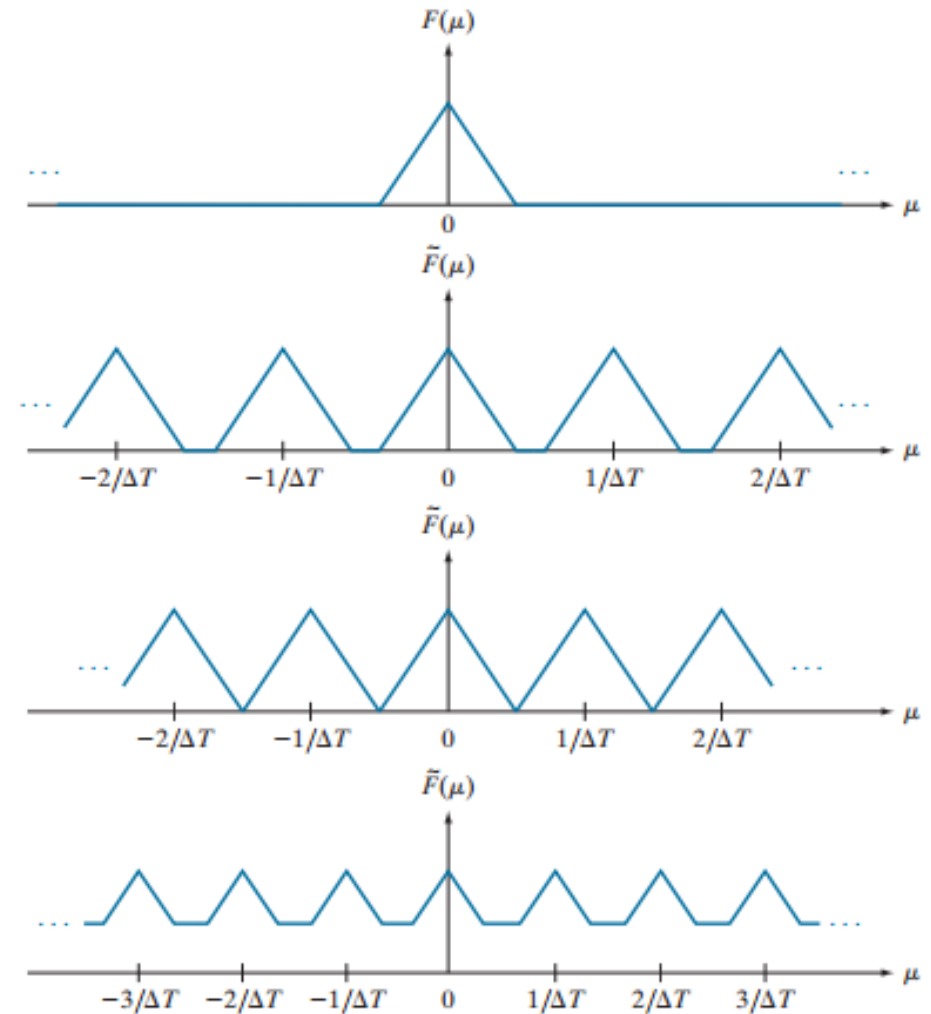


## ➤ The Fourier Transform of Sampled Functions

(a) Illustrative sketch of the Fourier transform of a band-limited function.

(b)–(d) Transforms of the corresponding sampled functions under the conditions of

**over-sampling**, **critically sampling**, and **under-sampling**, respectively.



## ➤ The Fourier Transform of Sampled Functions

➤ Extracting from  $\tilde{F}(\mu)$  a single period that is equal to  $F(\mu)$  is possible if the **separation** between copies is **sufficient**.

➤ sufficient separation is guaranteed if  $1/2\Delta T > \mu_{\max}$  or

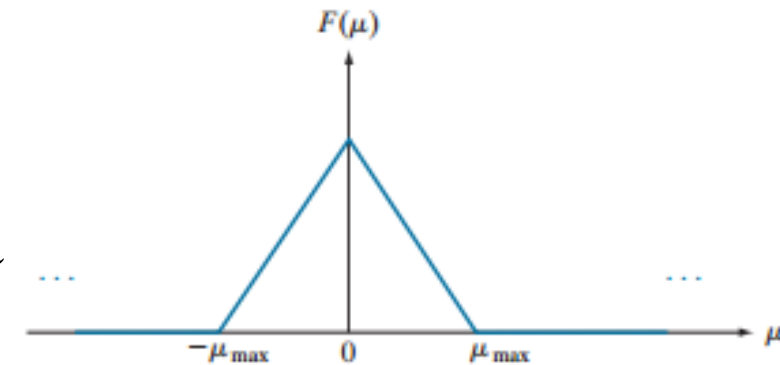
$$\frac{1}{\Delta T} > 2\mu_{\max}$$

➤ This equation indicates that a **continuous, band-limited function** can be recovered completely from a set of its samples if the samples are acquired at a rate **exceedingly twice** the **highest frequency** content of the function.

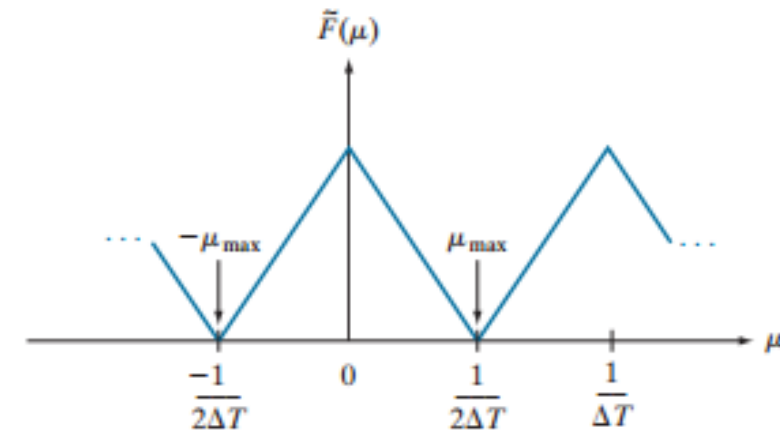
## ➤ The Sampling Theorem

- A function  $f(t)$  whose Fourier transform is zero for values of frequencies outside a finite interval (band)  $[-\mu_{\max}, \mu_{\max}]$  about the origin is called a **band-limited function**.

(a) Illustrative sketch of the Fourier transform of a band-limited function.

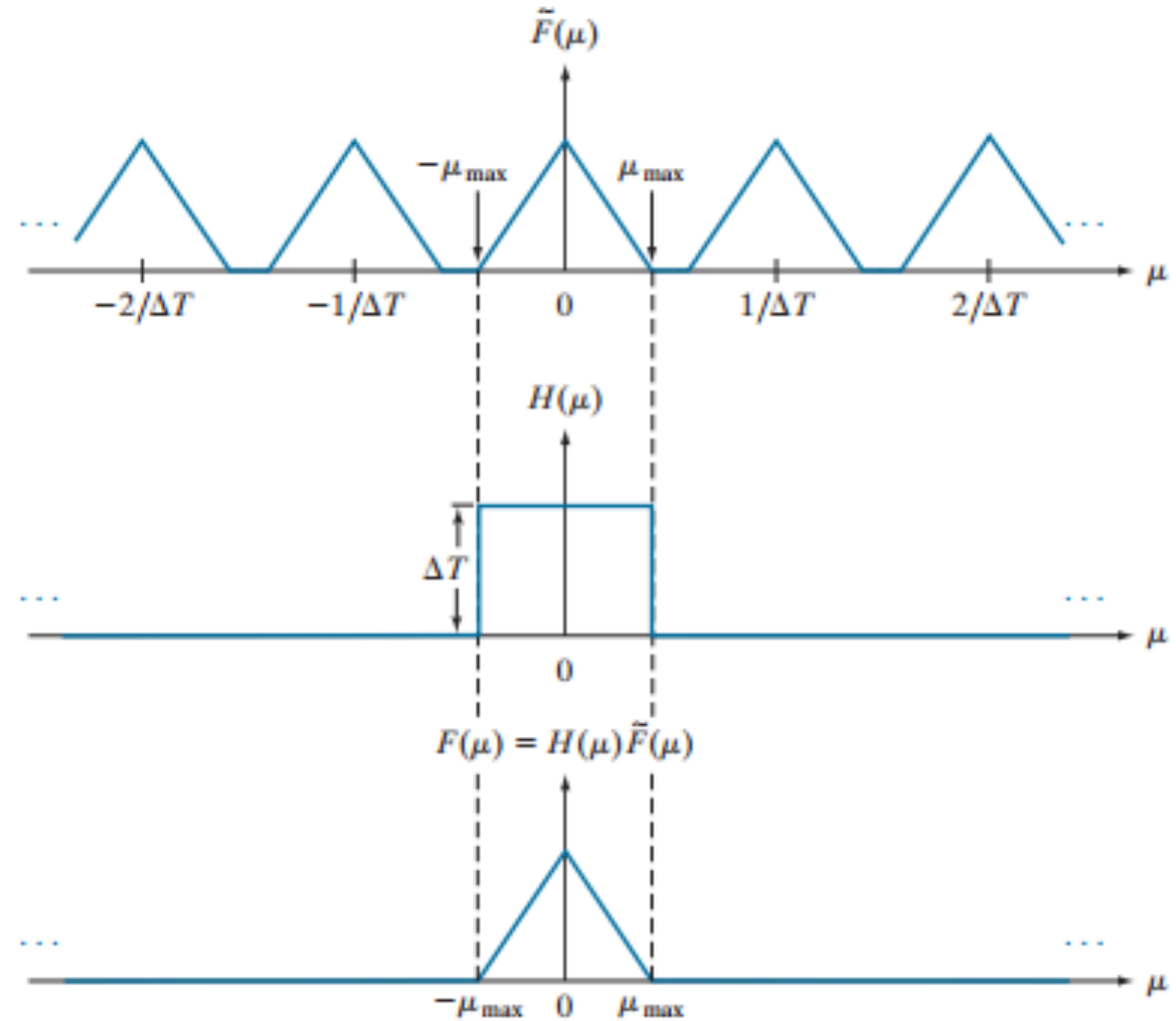


(b) Transform resulting from critically sampling that band-limited function.



## ➤ The Sampling Theorem

- (a) Fourier transform of a sampled, band-limited function.
- (b) Ideal lowpass filter transfer function.
- (c) The product of (b) and (a), used to extract one period of the infinitely periodic sequence in (a).



a  
b  
c

## ➤ Image Smoothing Using Lowpass Frequency Domain Filters

### Types of Lowpass Filters

```
graph TD; A[Types of Lowpass Filters] --> B[Ideal]; A --> C[Butterworth]; A --> D[Gaussian];
```

*Ideal*

*Butterworth*

*Gaussian*



## ➤ Types of Lowpass Filters

- These three categories cover the range from **very sharp** (**ideal**) to **very smooth** (**Gaussian**) filtering.
- The shape of a Butterworth filter is **controlled** by a parameter called the filter **order**.
- For **large values** of this parameter, the Butterworth filter approaches the **ideal filter**.
- For **lower values**, the Butterworth filter is more like a **Gaussian filter**.

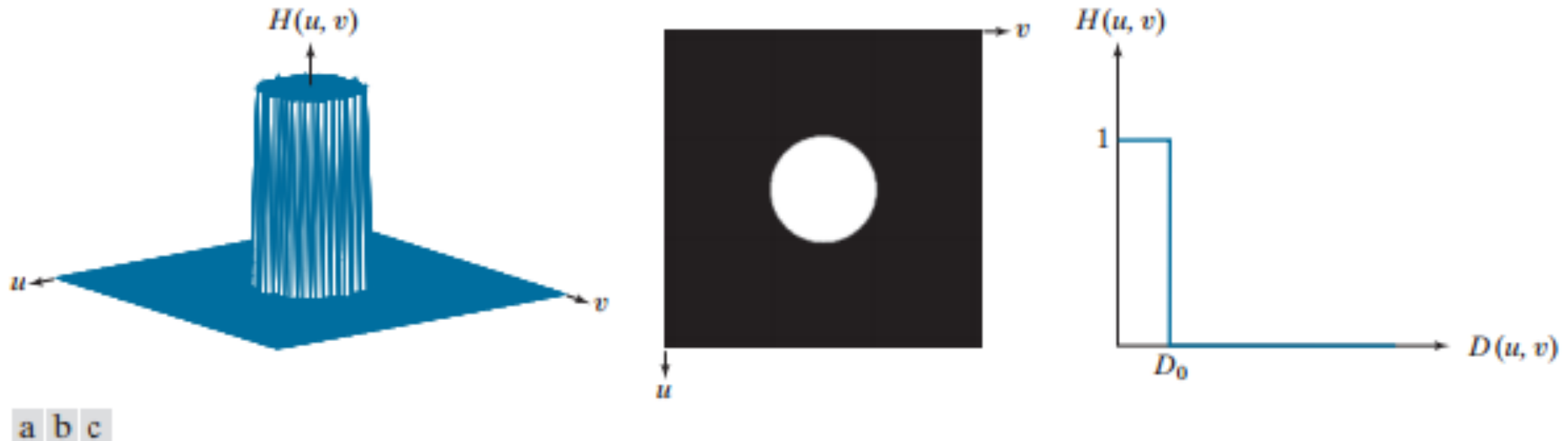
## ➤ Ideal Lowpass Filters

- A 2-D **lowpass filter** that passes without attenuation all frequencies within a circle of **radius** from the **origin**, and “**cuts off**” all frequencies outside this, circle is called an ideal lowpass filter (**ILPF**); it is specified by the transfer function

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) \geq D_0 \end{cases}$$

- where  $D_0$  is a positive constant, and  $D(u,v)$  is the distance between a point  $(u,v)$  in the frequency domain and the center of the  $P \times Q$  frequency rectangle;

## ➤ Ideal Lowpass Filters



- (a) Perspective plot of an ideal lowpass-filter transfer function.
- (b) Function displayed as an image.
- (c) Radial cross section.

## ➤ Ideal Lowpass Filters

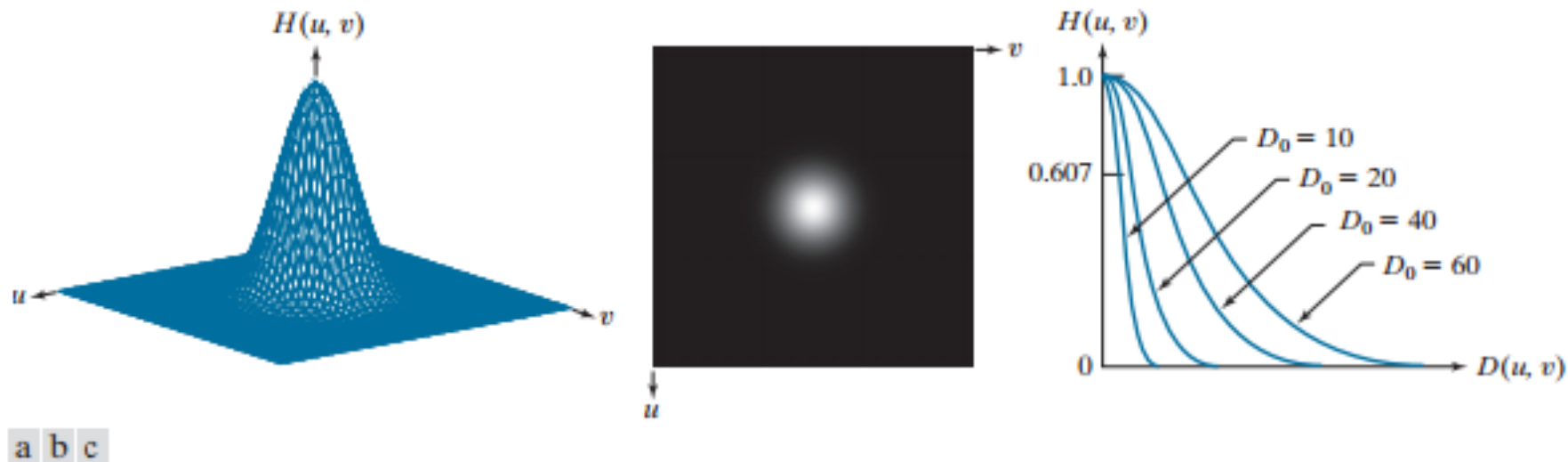


(a) Original image of size  $688 \times 688$  pixels. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460. The power removed by these filters was 13.1, 7.2, 4.9, 2.4, and 0.6% of the total, respectively. We used mirror padding to avoid the black borders characteristic of zero padding.

## ➤ Gaussian Lowpass Filters

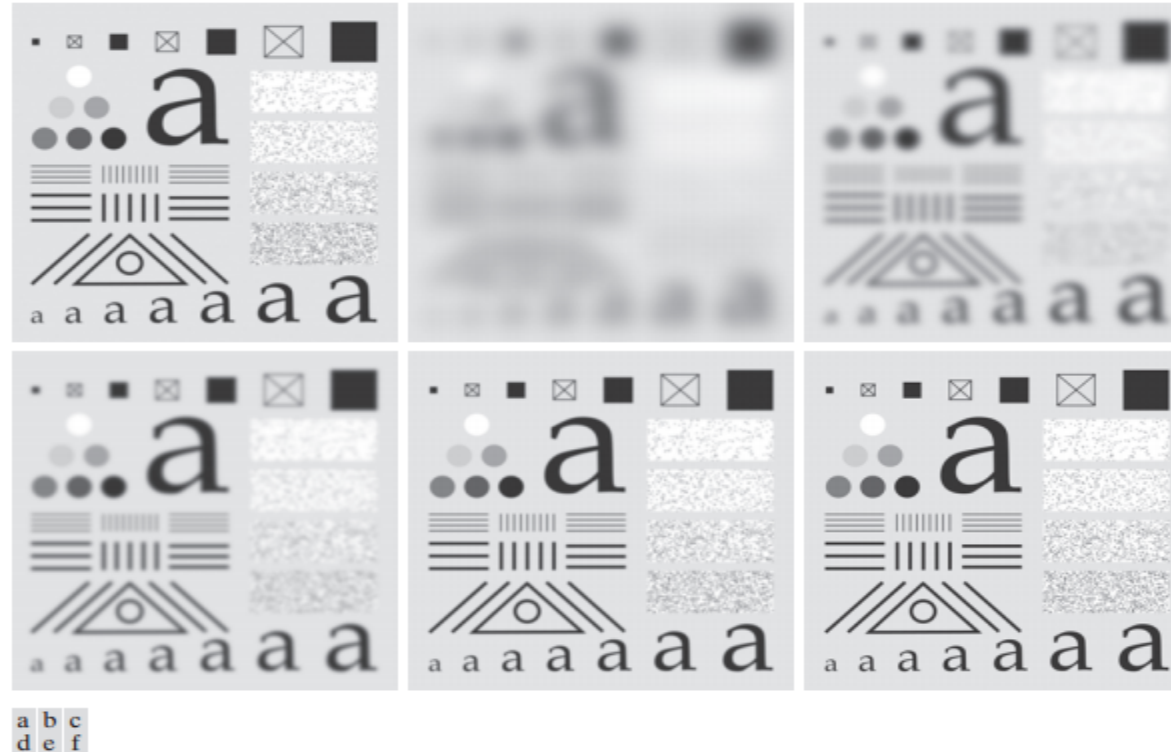
- Gaussian lowpass filter (GLPF) transfer functions have the form

$$H(u, v) = e^{-D^2(u,v)/2\sigma^2}$$



- (a) Perspective plot of a GLPF transfer function.
- (b) Function displayed as an image.
- (c) Radial cross sections for various values of  $D_0$ .

## ➤ Gaussian Lowpass Filters

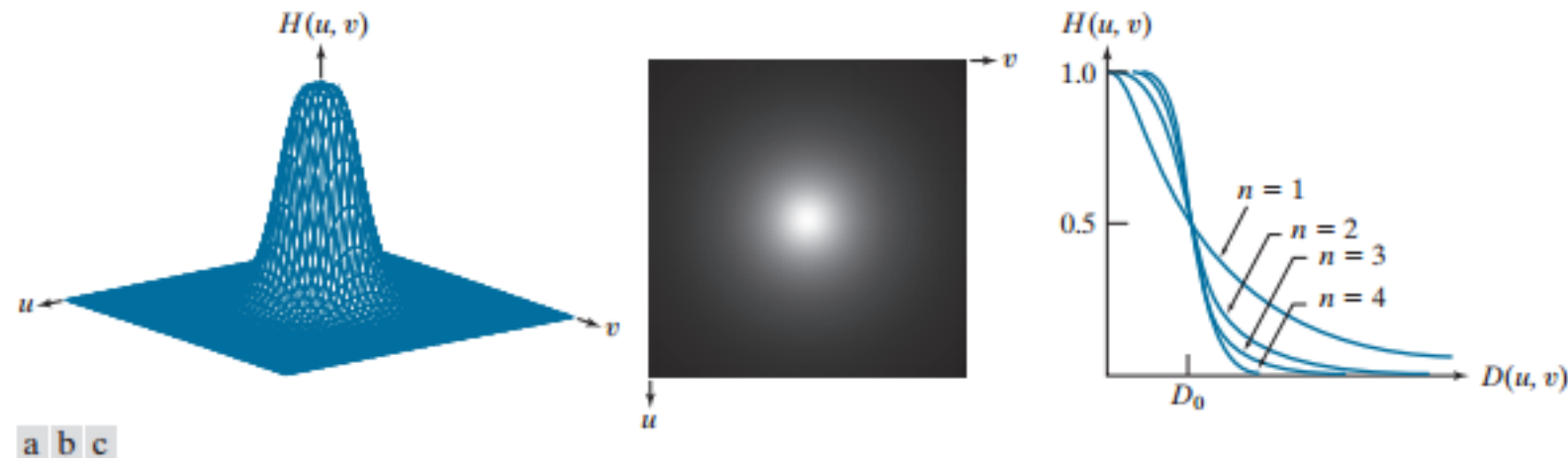


(a) Original image of size  $688 \times 688$  pixels. (b)–(f) Results of filtering using GLPFs with different cutoff frequencies. We used mirror padding to avoid the black borders characteristic of zero padding.

## ➤ Butterworth Lowpass Filters

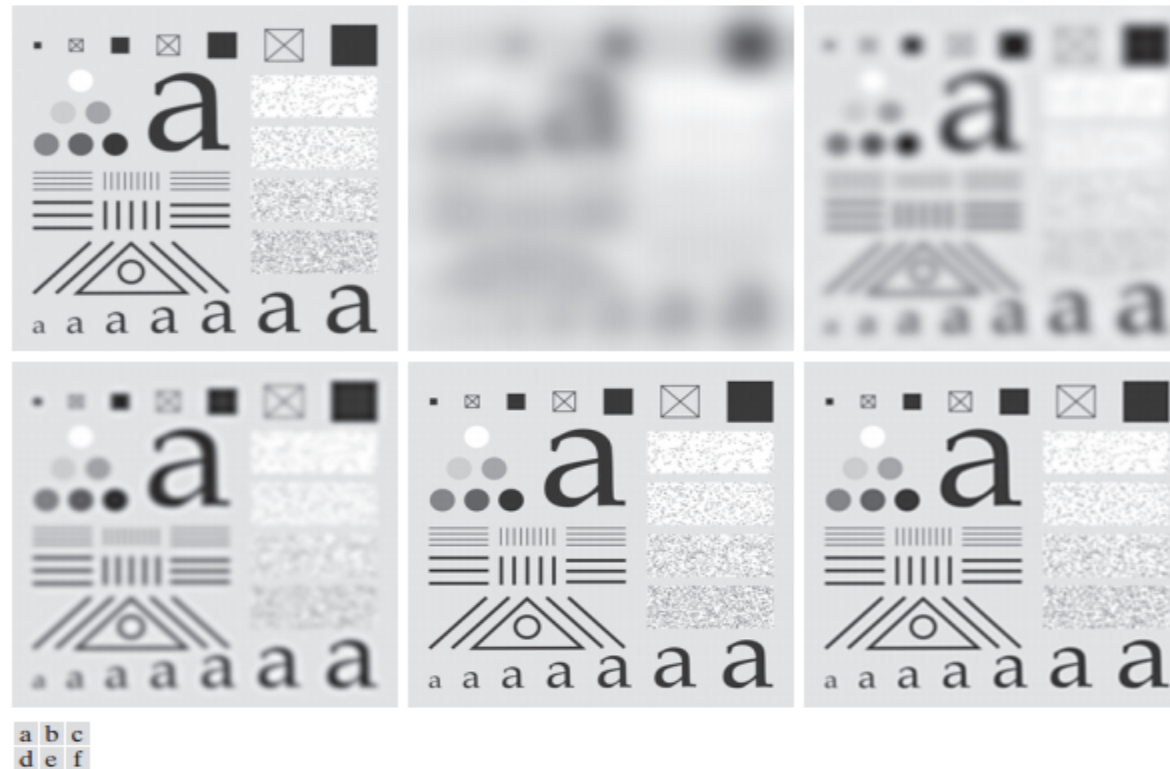
- The transfer function of a **Butterworth lowpass filter (BLPF)** of order  $n$ , with cutoff frequency at a distance  $D_0$  from the center of the frequency

rectangle, is defined as

$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$


(a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Function displayed as an image. (c) Radial cross sections of BLPFs of orders 1 through 4.

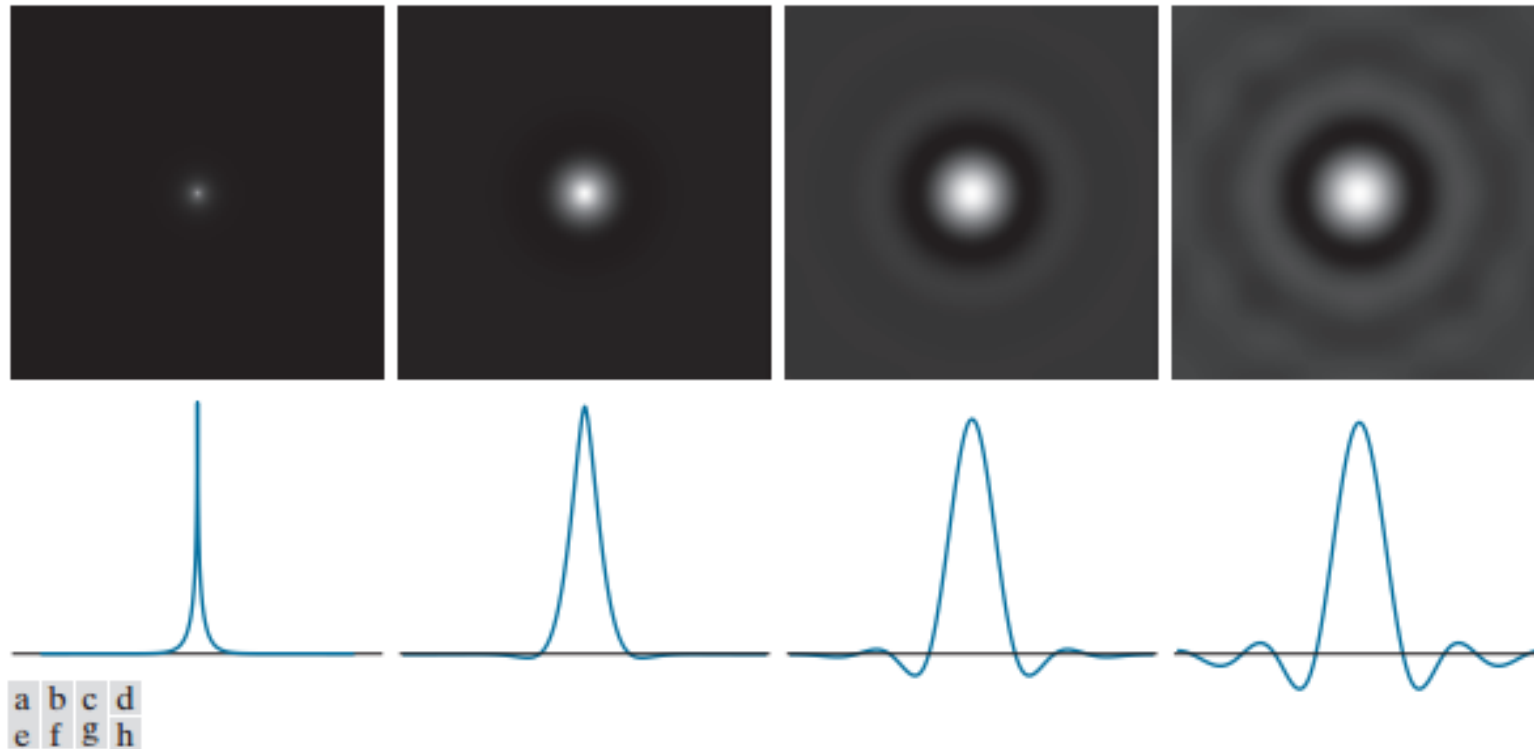
## ➤ Butterworth Lowpass Filters



(a) Original image of size  $688 \times 688$  pixels. (b)–(f) Results of filtering using BLPFs with cutoff frequencies at the different radii and  $n = 2.5$ . We used mirror padding to avoid the black borders characteristic of zero padding.



## ➤ Butterworth Lowpass Filters



(a)–(d) Spatial representations (i.e., spatial kernels) corresponding to BLPF transfer functions of size  $1000 \times 1000$  pixels, cut-off frequency of 5, and order 1, 2, 5, and 20, respectively.

(e)–(h) Corresponding intensity profiles through the center of the filter functions.

## ➤ Summary

- Lowpass filter transfer functions.  $D_0$  is the cutoff frequency, and  $n$  is the order of the Butterworth filter.

Ideal	Gaussian	Butterworth
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$	$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$

*Thank  
you*

